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Some hyperelliptic function identities that occur in the chiral Potts model

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Abstract. We present a number of hyperelliptic function identities that occur in the chiral Potts model. Some of these are relevant to the outstanding problem of solving the functional relations for the order parameter.

1. Introduction

The chiral Potts model is a model in statistical mechanics in which each site i of a planar (usually square) lattice contains a 'spin' σ_i that can take N values. Adjacent spins interact. This model was first studied in Hamiltonian form [1–3], and for small values of N [4, 5]. In 1988 the general-N integrable case of the lattice model was discovered [6]. This is 'solvable' in the sense that it satisfies the star–triangle (Yang–Baxter) relations, and the bulk free energy has been calculated. For $N > 2$ the order parameters have not been calculated, but a very believable and elegant conjecture has been made [7]. Its derivation remains a vexed problem [8], and is partly the motivation for this paper, where we present certain identities satisfied by the hyperelliptic functions that occur in the parametrization of the model and are relevant to the problem of solving the equations given in [8]. They have not all been proved; their status is explained in the summary.

As often happens, we have a conflict of notations: the vertical and horizontal rapidities of the chiral Potts model are usually denoted by p and q. We do this here, but as a result we write the nome of them hyperelliptic functions as x, so that q-series [9] become here x-series.

2. Hyperelliptic parametrization

The interactions along any edge are functions of N, of universal constants k and k', and of two 'rapidities' p and q. These are points in projective (a, b, c, d) space that lie on the curve

$$a^N + k'b^N = kd^N \qquad k'a^N + b^N = kc^N \tag{1}$$

where

$$k^2 + k'^2 = 1. \tag{2}$$

For $N = 2$ the model reduces to the Ising model solved by Onsager [10]; the curve is of genus 1, and can be parametrized in terms of ordinary single-variable Jacobi elliptic functions. For $N > 2$ the curve is of higher genus and we need to use hyperelliptic functions [11]. Define constants $\rho = \{\rho_1, \ldots, \rho_{N-1}\}$ in terms of k, k' as in [11], with $\rho_\alpha = \rho_{N-\alpha}$, $\rho_0 = \rho_N = 0$, and set

$$\tau_{\alpha\beta} = \rho_\alpha + \rho_\beta - \rho_{|\alpha-\beta|} \qquad \text{for } \alpha, \beta = 1, \ldots, N-1. \tag{3}$$

Let $s = \{s_1, \ldots, s_{N-1}\}$ be a set of variables. Define the hyperelliptic function

$$\Theta(s) = \sum_m \exp\left\{2\pi\mathrm{i}\sum_\alpha m_\alpha s_\alpha + \pi\mathrm{i}\sum_\alpha\sum_\beta m_\alpha \tau_{\alpha\beta} m_\beta\right\} \tag{4}$$

the inner sums being over $\alpha, \beta = 1, \ldots, N-1$, and the outer sum over all values of the integers $m = \{m_1, \ldots, m_{N-1}\}$.

Let $g = \{g_1, \ldots, g_{N-1}\}$, where

$$g_\alpha = \alpha/N \tag{5}$$

write $\{g_1/2, \ldots, g_{N-1}/2\}$ in the obvious way as $g/2$, and similarly for s, ρ. Define

$$k = \sin\theta \qquad k' = \cos\theta \qquad \omega = \mathrm{e}^{2\pi\mathrm{i}/N} \tag{6}$$

$$\chi = \mathrm{i}\mathrm{e}^{\mathrm{i}(2-N)\theta/N} \qquad \langle s\rangle = (s_1 + \cdots s_{N-1})/N. \tag{7}$$

Then from equation (48) of [11], we can take $a, \ldots, d$ to be

$$\begin{aligned} a &= \mathrm{e}^{\mathrm{i}\pi/2N}\mathrm{e}^{2\pi\mathrm{i}\langle s\rangle}\Theta(s - \tfrac{1}{2}g + \tfrac{1}{2}\rho) \qquad b = \chi\mathrm{e}^{\mathrm{i}\pi/N}\Theta(s - \tfrac{1}{2}g - \tfrac{1}{2}\rho) \\ c &= \Theta(s + \tfrac{1}{2}g - \tfrac{1}{2}\rho) \qquad d = \chi\mathrm{e}^{-\mathrm{i}\pi/2N}\mathrm{e}^{2\pi\mathrm{i}\langle s\rangle}\Theta(s + \tfrac{1}{2}g + \tfrac{1}{2}\rho). \end{aligned} \tag{8}$$

Let

$$\begin{aligned} D_{\alpha\lambda} &= 0 \qquad \text{if } \alpha < \lambda \\ &= 1 \qquad \text{if} \alpha \geqslant \lambda \end{aligned} \tag{9}$$

and write $\Theta(s)$ alternatively as $\Theta[s]$ or $\Theta\{s_\alpha\}$. Then $s_1, \ldots s_{N-1}$ are not independent, but must satisfy the $N-2$ relations

$$\Theta\{s_\alpha + \tfrac{1}{2}\tau_{\alpha\lambda} - \tfrac{1}{2}D_{\alpha\lambda}\} = 0 \qquad \text{for } \lambda = 2, \ldots, N-1 \tag{10}$$

leaving one degree of freedom for $s_1, \ldots, s_{N-1}$.

A related hyperelliptic function $\Theta_1(s)$ is obtained by replacing each s_α in (4) by $s_\alpha - \frac{1}{2}$, and letting m_1 range over all half-an-odd-integer values ($m_2, \ldots, m_{N-1}$ remain integers). From equation (32) of [11],

$$\Theta_1(s) = -\mathrm{i}\exp\{\pi\mathrm{i}(s_1 + \tfrac{1}{2}\rho_1)\}\Theta\{s_\alpha + \tfrac{1}{2}\tau_{\alpha 1} - \tfrac{1}{2}D_{\alpha 1}\}. \tag{11}$$

It is an odd function, zero when all its arguments are zero:

$$\Theta_1(-s) = -\Theta_1(s) \qquad \Theta_1(0) = 0. \tag{12}$$

2.1. Invariances

The Θ-function has the following symmetries (true for arbitrary $s_1, \ldots, s_{N-1}$, not necessarily satisfying (10)):

$$\begin{aligned} &\Theta\{s_\alpha + \delta_{\alpha j}\} = \Theta(s) = \Theta(-s) \\ &\Theta\{s_\alpha + \tau_{\alpha j}\} = \exp\{-2\pi\mathrm{i}(s_j + \rho_j)\}\Theta(s) \\ &\Theta(s) = \Theta\{s_{\alpha-1} - s_{N-1}\} = \Theta\{s_{N+1-\alpha} - s_1\} \end{aligned} \tag{13}$$

for $j = 1, \ldots, N-1$, taking $s_0 = s_N = 0$. Hence

$$\begin{aligned} \Theta_1\{s_\alpha + \delta_{\alpha j}\} &= \exp\{\pi \mathrm{i} D_{1j}\}\Theta_1(s) \\ \Theta_1\{s_\alpha + \tau_{\alpha j}\} &= -\exp\{-2\pi\mathrm{i}(s_j + \rho_j)\}\Theta_1(s). \end{aligned} \tag{14}$$

Using the symmetries (13) (with $j = 1$), for all integers n and arbitrary $s_1, \ldots, s_{N-1}$ we can verify that

$$\Theta\{n\rho_\alpha + s_\alpha\} = \mathrm{e}^{-2\pi \mathrm{i} n s_1}\Theta\{n\rho_\alpha + s_{\alpha+1} - s_1\}.$$

Setting $s_\alpha = j\alpha/N$ for $\alpha = 1, \ldots, N-1$, with j an integer, it follows that

$$\Theta(n\rho + jg) = 0 \text{ provided } nj/N \text{ is not an integer.} \tag{15}$$

Numerical calculations suggest that (15) may also be true when $N = 4$ and $n = j = 2$, which in turn suggests that it may be true for the more general situation when neither n nor j is divisible by N. This would be consistent with the zeros of our later equations (26), (27), but we have not proved this. Its proof must be more subtle: (13)–(15) are true for arbitrary $\rho_1, \ldots, \rho_{N-1}$ (with $\rho_j = \rho_{N-j}$), whereas a leading-order calculation for small k' indicates that, for $N = 4$, $\Theta(2\rho + 2g)$ is zero only when ρ_1, ρ_2 are related as in [11].

There are various automorphisms that take a, b, c, d, s_α to the values indicated in the folowing table, leaving the relations between them unchanged, namely

$M_j^{(1)}$	ωa	b	c	ωd	$s_\alpha + \delta_{\alpha j}$
$M_j^{(2)}$	$\omega^j a$	$\omega^j b$	c	d	$s_\alpha + \tau_{\alpha j}$
$M^{(3)}$	c	$\omega^{1/2} d$	$\omega^{-1/2} a$	$\omega^{-1} b$	$-s_\alpha$
$M^{(4)}$	a	b	$\omega^{-1} c$	d	$s_{\alpha-1} - s_{N-1} + \frac{1}{2}(\delta_{\alpha 1} + \tau_{\alpha 2} - \tau_{\alpha 1})$
$M^{(5)}$	d	$\omega^{1/2} c$	$\omega^{-1/2} b$	a	$s_{N+1-\alpha} - s_1 - \frac{1}{2}\tau_{\alpha 1}$
R	b	ωa	d	c	$s_1 - s_{N+1-\alpha} + \frac{1}{2}\tau_{\alpha 1}$

for $j = 1, \ldots, N-1$, ignoring overall normalization factors of a, b, c, d. For instance, the mapping $M^{(5)}$ takes b to $\omega^{1/2}c$, and s_α to $s_{N+1-\alpha} - s_1 - \frac{1}{2}\tau_{\alpha 1}$. Note that $R = M^{(3)}M^{(5)}$, $R^2 = M_1^{(2)}$ and ${M^{(3)}}^2 = {M^{(5)}}^2 = 1$.

2.2. A basic identity

The variables $a, b, c, d, s = \{s_1, \ldots, s_{N-1}\}$ are all functions of the rapidity p (put another way, p is specified by these variables, or by some subset of them), so should bear an index p. Let us write them as $a_p, b_p, c_p, d_p, s_p = \{(s_p)_1, \ldots, (s_p)_{N-1}\}$, with the convention that a Greek, numerical, N or $N-1$ suffix denotes the element of the set s, while a lower case Roman suffix (p, q, r or t) denotes the rapidity of the whole set. A related variable, in terms of which we originally obtained the hyperelliptic function parametrization is v_p, where

$$\frac{b_p d_p}{a_p c_p} = \exp\{\mathrm{i}(\pi + 2v_p)/N\}. \tag{16}$$

Then for any four rapidities p, q, r, t

$$\begin{aligned} &\Theta_1(s_p - s_q)\Theta_1(s_p + s_q)\Theta_1(s_r - s_t)\Theta_1(s_r + s_t) \\ &\qquad -\Theta_1(s_p - s_t)\Theta_1(s_p + s_t)\Theta_1(s_r - s_q)\Theta_1(s_r + s_q) \\ &\quad = \Theta_1(s_r - s_p)\Theta_1(s_r + s_p)\Theta_1(s_t - s_q)\Theta_1(s_t + s_q). \end{aligned} \tag{17}$$

This is a very general identity which extends the corresponding identity for ordinary elliptic functions: equation (15.3.10) of [12]. It can be proved in much the same way.

Consider the ratio of the LHS to the RHS: from (12) and (14), it is invariant under the automorphisms $M_j^{(1)}$, $M_j^{(2)}$, $M^{(3)}$, applied to the rapidity p (say). It follows that it is a single-valued function of the variable v_p (it is independent of the choice of the contour and the sign of Δ in equation (35) of [11]). It is analytic, except possibly for isolated singularities at the zeros of the RHS, and at the points where $(s_p)_1, \ldots, (s_p)_{N-1}$ are nonanalytic functions of v_p. These last are when v_p equals $\pm\theta$ or ∞, and it can be verified that the ratio remains finite as v_p approaches these values. Since it is analytic in their neighbourhood, it is therefore analytic at these points.

There are two possible types of zero of the RHS: there are those whose location is dependent on the other rapidity r, occurring when $s_p = \pm s_r$, to within the automorphisms $M_j^{(1)}$, $M_j^{(2)}$, i.e. when $v_p = v_r$. They are simple zeros, and it is obvious that the LHS then also vanishes, so the ratio is analytic.

The other possible zeros are when

$$(s_p)_\alpha = \tfrac{1}{2}(\tau_{\alpha 1} - \tau_{\alpha j} + D_{\alpha j} - D_{\alpha 1}) \tag{18}$$

for $j = 2, \ldots, N-1$. If this happens, then from (11), the factor $\Theta_1(s_r - s_p)$ is proportional to $\Theta\{(s_r)_\alpha + \frac{1}{2}\tau_{\alpha j} - \frac{1}{2}D_{\alpha j}\}$, which vanishes because of the restrictions (10). However, the same is also true of the factors $\Theta_1(s_p - s_q)$ and $\Theta_1(s_p - s_t)$ on the RHS (and, because of the $M^{(3)}$ automorphism, of the factors $\Theta_1(s_r + s_p)$, $\Theta_1(s_p + s_q)$, $\Theta_1(s_p + s_t)$).

One should consider whether the values (18) of $(s_p)_1, \ldots, (s_p)_{N-1}$ are attained for some value of v_p, i.e. whether they satisfy (10). For $N = 3$ they do, but for $N > 3$ it seems that they do not. In either case there is no problem: if they do attain (18) then each of the six factors will have a simple zero in v_p, and these factors will cancel out of the ratio of the LHS to the RHS of (17). If they do not, then there are no such zeros to cancel.

In either case,the ratio is analytic for all v_p (including the point at infinity), so from Liouville's theorem it is a constant. Taking $v_p = v_t$, we can choose $s_p = s_t$ and observe that this constant is unity. This establishes the identity (17).

We shall present a number of further identities; first we need some more definitions:

$$t_\alpha = s_1 + \cdots + s_\alpha - \alpha(s_1 + \cdots + s_{N-1})/N. \tag{19}$$

As with s, let $t = \{t_1, \ldots, t_{N-1}\}$, and write t_p for the set t corresponding to the rapidity p. The automorphism $R^2 = M_1^{(2)}$ applied to p takes t_p to $t_p + \rho$, while $M_1^{(1)} \ldots M_{N-1}^{(1)}$ takes t_p to $t_p + g$.

3. Functions of two rapidity variables

The Boltzmann weights of the chiral Potts model are

$$W_{pq}(n) = \prod_{j=1}^{n} \frac{d_p b_q - a_p c_q \omega^j}{b_p d_q - c_p a_q \omega^j} \qquad \overline{W}_{pq}(n) = \prod_{j=1}^{n} \frac{\omega a_p d_q - d_p a_q \omega^j}{c_p b_q - b_p c_q \omega^j}. \tag{20}$$

Both are periodic of period N: $W_{pq}(n+N) = W_{pq}(n)$, $\overline{W}_{pq}(n+N) = \overline{W}_{pq}(n)$; regarding $\overline{W}_{pq}(i-j)$ as the element i, j of an $N \times N$ cyclic matrix $\overline{W}_{pq}$, a quantity that occurs in the functional relations for the transfer matrices [13] is

$$f_{pq} = (\det \overline{W}_{pq})^{1/N} \Big/ \prod_{n=0}^{N-1} W_{pq}(n)^{1/N}. \tag{21}$$

Another quantity that occurs in the functional relations for the generalized correlation function [8] is

$$L_{pq}^{(0)}(j) = k^{(N-1)/N} \prod_{m=1}^{N-1} \left\{ \frac{b_p d_q - \omega^{j-m-1} d_p b_q}{c_p d_q - \omega^{j-m-1} d_p c_q} \frac{c_p a_q - \omega^{j-m-1} a_p c_q}{b_p a_q - \omega^{j-m-1} a_p b_q} \right\}^{m/N} . \tag{22}$$

3.1. Some general-N identities

With the normalization of equation (8), it follows from (17) that

$$a_q c_q b_p d_p - a_p c_p b_q d_q = -\chi^2 \exp\left\{ \frac{\mathrm{i}\pi}{N} + 2\pi\mathrm{i}\langle s_p\rangle + 2\pi\mathrm{i}\langle s_q\rangle \right\} \times \Theta_1(\rho)\Theta_1(g)\Theta_1(s_q - s_p)\Theta_1(s_q + s_p). \tag{23}$$

Two identities that we conjecture to be true, but have not proved, are

$$\prod_{n=1}^{N-1} \overline{W}_{pq}(n) = \exp\left\{ \pi\mathrm{i} \sum_{\alpha=1}^{N-1} (2\alpha - N - 1)(s_q - s_p - \rho)_\alpha \right\} \prod_{j=0}^{N-1} \frac{\Theta(t_p - t_q + jg + \rho)}{\Theta(t_p - t_q + jg)} \tag{24}$$

$$L_{pq}^{(0)}(j) = \frac{\Theta(t_q - t_p + jg)}{\Theta(t_q - t_p + (j-1)g)}. \tag{25}$$

Remembering that $t_{R^2 q} = t_q + \rho$ and forming the ratio $L_{p,R^2 q}^{(0)}(j)/L_{pq}^{(0)}(j)$, from (25) we obtain

$$\frac{\omega c_p a_q - \omega^j a_p c_q}{b_p d_q - \omega^j d_p b_q} = \frac{\Theta[t_q - t_p + (j-1)g]\Theta(t_q - t_p + jg + \rho)}{\Theta(t_q - t_p + jg)\Theta[t_q - t_p + (j-1)g + \rho]} \tag{26}$$

while the ratio $L_{p,q}^{(0)}(j+1)/L_{pq}^{(0)}(j)$ gives

$$\frac{b_p d_q - \omega^j d_p b_q}{c_p d_q - \omega^j d_p c_q} \frac{c_p a_q - \omega^j a_p c_q}{b_p a_q - \omega^j a_p b_q} = k^{-2/N} \frac{\Theta(t_q - t_p + jg)^2}{\Theta[t_q - t_p + (j-1)g]\Theta[t_q - t_p + (j+1)g]} \tag{27}$$

from which one can deduce that

$$\Theta(jg)/\Theta(0) = k^{j(N-j)/N} \qquad \text{for } j = 0, \ldots, N. \tag{28}$$

4. The case $N = 3$

If $N = 3$, then $\rho_1 = \rho_2 = \rho$, so if we define

$$x = \mathrm{e}^{2\pi\mathrm{i}\rho} \qquad |x| < 1 \tag{29}$$

then

$$\Theta\{s_1, s_2\} = \Phi(\mathrm{e}^{2\pi\mathrm{i}s_1}, \mathrm{e}^{2\pi\mathrm{i}s_2}) \tag{30}$$

where

$$\Phi(\alpha, \beta) = \sum_{m,n} x^{m^2+mn+n^2} \alpha^m \beta^n \tag{31}$$

the sum being over all integers m, n.

4.1. Identities for arbitrary arguments

First we prove some identities that are true for all arguments α, β of the hyperelliptic functions.

The function $\Phi(\alpha, \beta)$ can be expressed as the sum of two products of single-variable Θ-functions [11]: for this reason such single variable functions occur in the following relations. Four that we shall need are

$$\tilde{\psi}(z) = \prod_{n=1}^{\infty}(1 - x^{3n-2}z)(1 - x^{3n-1}/z) \qquad \psi(z) = z^{-1/6}\tilde{\psi}(z) \tag{32}$$

$$r(z) = z^{-1/2}\prod_{n=1}^{\infty}(1 - x^{n-1}z)(1 - x^{n}/z) \qquad \phi(z) = \psi(z^{-1})/\psi(z). \tag{33}$$

The symmetry and quasiperiodicity relations (13) become

$$\Phi(\alpha, \beta) = \Phi(\beta, \alpha) = \Phi(\alpha^{-1}, \beta^{-1}) = \Phi(\alpha, \alpha/\beta) = \Phi(\beta/\alpha, \beta) \tag{34}$$

$$\Phi(x^2\alpha, x\beta) = x^{-1}\alpha^{-1}\Phi(\alpha, \beta) \qquad \Phi(x\alpha, x^2\beta) = x^{-1}\beta^{-1}\Phi(\alpha, \beta) \tag{35}$$

and hence $\Phi(x^3\alpha, \beta) = x^{-3}\beta\alpha^{-2}\Phi(\alpha, \beta)$, $\Phi(\alpha, x^3\beta) = x^{-3}\alpha\beta^{-2}\Phi(\alpha, \beta)$.

From these one can deduce that

$$\Phi(\alpha/x, \beta) = \alpha\Phi(x^{-1}\beta^{-1}, \alpha/\beta). \tag{36}$$

Setting $\alpha = 1/\beta = \omega$ or ω^2, it follows that

$$\Phi(\omega/x, \omega^2) = \Phi(\omega^2/x, \omega) = 0 \tag{37}$$

which is a special case of (15).

Three related functions are

$$\Phi_j(\alpha, \beta) = \sum_{m-n=j,\ \mathrm{mod}\ 3} x^{m^2+mn+n^2}\alpha^m\beta^n \tag{38}$$

the sum being over all integers m, n such that $m - n = j$, modulo 3. Then

$$\Phi(\omega^j\alpha, \omega^{-j}\beta) = \Phi_0(\alpha, \beta) + \omega^j\Phi_1(\alpha, \beta) + \omega^{2j}\Phi_2(\alpha, \beta) \tag{39}$$

and

$$\Phi_j(x\alpha, x^2\beta) = x^{-1}\beta^{-1}\Phi_{j-1}(\alpha, \beta). \tag{40}$$

4.2. First product–sum relation

Let α, β, u be three variables related by

$$\alpha = u^3\beta^2 \tag{41}$$

and consider the product

$$h(\alpha, \beta) = \Phi(u, u^2\beta)\Phi(\omega u, \omega^2u^2\beta)\Phi(\omega^2u, \omega u^2\beta). \tag{42}$$

(This is the function $\prod\Theta(t + jg)$ that occurs in (24).)

The RHS is an analytic function of u and β in the domain $0 < |u|, |\beta| < \infty$, and is unchanged by replacing u by ωu, so has a convergent double Laurent expansion in powers of u^3 and β, or equivalently of α and β:

$$h(\alpha, \beta) = \sum_{m,n} c_{mn}\alpha^m\beta^n. \tag{43}$$

It satisfies the quasiperiodicity relations

$$h(x^3\alpha,\beta) = x^{-3}\beta\alpha^{-2}h(\alpha,\beta) \qquad h(\alpha,x^3\beta) = x^{-3}\alpha\beta^{-2}h(\alpha,\beta). \tag{44}$$

The functions Φ, Φ_0, Φ_1, Φ_2 also satisfy these, but h does *not* satisfy the stronger relations (35).

Substituting the expansion (43) into these relations and equating coefficients:

$$c_{m+2,n-1} = x^{3m+3}c_{mn} \qquad c_{m-1,n+2} = x^{3n+3}c_{mn}. \tag{45}$$

The general solution of these equations is $c_{mn} = d_{m-n}x^{m^2+mn+n^2}$, where $d_{m+3} = d_m$, so $h(\alpha,\beta)$ must be a linear combination of $\Phi_0(\alpha,\beta)$, $\Phi_1(\alpha,\beta)$, $\Phi_2(\alpha,\beta)$:

$$h(\alpha,\beta) = C_0\Phi_0(\alpha,\beta) + C_1\Phi_1(\alpha,\beta) + C_2\Phi_2(\alpha,\beta) \tag{46}$$

the coefficients C_0, C_1, C_2 depending only on x.

Inverting α, β, u in (42), and using the symmetry $\Phi(\alpha^{-1},\beta^{-1}) = \Phi(\alpha,\beta)$, we find that $h(\alpha^{-1},\beta^{-1}) = h(\alpha,\beta)$. Since $\Phi_1(\alpha^{-1},\beta^{-1}) = \Phi_2(\alpha,\beta)$, this implies that $C_1 = C_2$. Setting $\alpha, \beta, u = x, x^2, x^{-1}$ in (42) and using (37), we note that

$$h(x,x^2) = 0. \tag{47}$$

Substituting this into (46) and using (40), it follows that

$$C_0\Phi_2(1,1) + C_1[\Phi_0(1,1) + \Phi_1(1,1)] = 0. \tag{48}$$

Let

$$V(x) = \Phi(1,1) \qquad V_j(x) = \Phi_j(1,1). \tag{49}$$

We note at this point that we are treading in the footsteps of the masters: the function $V(x)$ was examined by Ramanujan, who wrote down the formula

$$V(x) = 1 + 6\sum_{n=1}^{\infty}\left\{\frac{x^{3n-2}}{1-x^{3n-2}} - \frac{x^{3n-1}}{1-x^{3n-1}}\right\}. \tag{50}$$

From (39) and symmetry,

$$V(x) = V_0(x) + V_1(x) + V_2(x) \qquad V_1(x) = V_2(x). \tag{51}$$

Expanding these functions, we find to the first few terms that

$$\begin{aligned} V_0(x) &= V(x^3) = 1 + 6x^3 + 6x^9 + 6x^{12} + 12x^{21} + \cdots \\ V_1(x) &= V_2(x) = 3x + 3x^4 + 6x^7 + 6x^{13} + \cdots \end{aligned} \tag{52}$$

which led us to conjecture that

$$\begin{aligned} &V_0(x) = V(x^3) \qquad V_1(x) = V_2(x) = 3xQ(x^9)^3/Q(x^3) \\ &V_0(x) - V_1(x) = Q(x)^3/Q(x^3) \qquad V_0(x)^3 - V_1(x)^3 = Q(x^3)^9/Q(x^9)^3 \end{aligned} \tag{53}$$

where

$$Q(x) = \prod_{n=1}^{\infty}(1-x^n). \tag{54}$$

The author is indebted to George E Andrews [14] and Michael Hirschhorn [15] for providing proofs of these relations (53). In particular, Andrews showed that they are simple consequences of the identities proved by Borwein *et al* [16] in their paper on the cubic modular identities of Ramanujan. This is shown in the appendix.

Solving (48) for C_1/C_0 and obtaining the normalization of C_0, C_1, C_2 by setting $\alpha = \beta = 1$ in (46), we obtain the alternative expression for $h(\alpha,\beta)$:

$$h(\alpha,\beta) = [Q(x)^3/Q(x^3)][V(x)\Phi_0(\alpha,\beta) - V_1(x)\Phi(\alpha,\beta)]. \tag{55}$$

4.3. Second product–sum relation

A related identity that we shall use is proposition 2.2 of [16], namely

$$\Phi(x,x) = 3Q(x^3)^3/Q(x). \tag{56}$$

(One can verify directly from their definitions as double sums that $V_1(x) = x\Phi(x^3,x^3)$ [15].) From (34) and (37), $\Phi(x,x) = \Phi(1,x)$ and $\Phi(\omega,\omega^2 x) = \Phi(\omega^2,\omega x) = 0$.

It follows that

$$\Phi(u,x/u) = Q(x)Q(x^3)\prod_{n=1}^{\infty}(1-\omega x^{n-1}u)(1-\omega^2 x^n u^{-1})(1-\omega^2 x^{n-1}u)(1-\omega x^n u^{-1}). \tag{57}$$

The proof of (57) is similar to (but simpler than) that of (55). Writing either side as $f(u)$, each satisfies the quasiperiodicity relation $f(xu) = u^{-2}f(u)$, and is Laurent expandable in the domain $0 < |u| < \infty$. Their difference therefore has a Laurent expansion $\sum c_n u^n$, where $c_{n+2} = x^n c_n$. Thus c_n is proportional to c_0 (c_1) if n is even (odd), so the difference is a combination of just two linearly independent functions, with coefficients c_0 and c_1. Both sides vanish when $u = \omega$ (or when $u = \omega^2$), which fixes the ratio c_1/c_0. From (56), the two sides are equal when $u = 1$, which fixes c_0, c_1 to be zero and proves (57).

Just as we used (56) to establish the more general relation (57), so we can use (57) to establish that

$$\begin{aligned}\tilde{\psi}(\alpha)\tilde{\psi}(\beta)\tilde{\psi}(\alpha/\beta) &+ u^2\beta\tilde{\psi}(1/\alpha)\tilde{\psi}(\beta)\tilde{\psi}(\beta/\alpha) + u\beta\tilde{\psi}(1/\alpha)\tilde{\psi}(1/\beta)\tilde{\psi}(\alpha/\beta)\\ &= Q(x)^{-2}Q(x^3)^{-4}\Phi(u,u^2\beta/x)\Phi(\omega^{-1}u,\omega u^2\beta)\Phi(\omega u,\omega^{-1}u^2\beta)\end{aligned} \tag{58}$$

for all α,β,u satisfying (41).

To prove this, regard u,β as independent complex variables and α as defined by (41). Write either side as $f(u,\beta)$. Then each satisfies the quasiperiodicity and symmetry relations

$$\begin{aligned}f(u,\beta) &= (x^2\alpha^2/\beta)f(xu,\beta) = x^2\alpha\beta f(u/x,x^3\beta)\\ f(u,\beta) &= f(u^{-1},u^3\beta) = u^2\beta f(u\beta,u^{-3}\beta^{-2}).\end{aligned} \tag{59}$$

Each is Laurent expandable in integer powers of u and β: it follows that the difference can be written as

$$\sum_{mn} x^{m^2/3-mn+n^2-n/3}u^m\beta^n$$

where $c_{mn} = c_{m+3,n} = c_{m,n+3} = c_{3n-m,n} = c_{3n-2m+1,n-m+1}$. The c_{mn} are therefore all linear combinations of c_{00}, c_{01}, c_{02}, so the difference is the sum of just three linearly independent functions of u and β, with coefficients c_{00}, c_{01}, c_{02}.

If we choose $\beta = x^2/u^3$, then $\alpha/\beta = x^2$, $\tilde{\psi}(\alpha/\beta)$ vanishes and we regain (57). The difference therefore vanishes. The three functions now depend only on u, but are still linearly independent (to leading order they are proportional to $u^2+u^3+u^4+u^6+u^7+u^8$, $u^3+u^5+u^7$, $u^4+u^5+u^6$), so c_{00}, c_{01}, c_{02} must all be zero. Hence (58) is true for arbitrary u, β.

4.4. Identities involving k, p, q

A number of hyperelliptic function identities for the three-state chiral Potts model have already been obtained [17–19]. The relation between k and x is

$$(k'/k)^2 = 27x[Q(x^3)/Q(x)]^{12} \tag{60}$$

and if we define

$$\begin{aligned} \alpha &= \alpha_{pq} = \exp\{2\pi \mathrm{i}[(s_q)_1 - (s_p)_1]\} \\ \beta &= \beta_{pq} = \exp\{2\pi \mathrm{i}[(s_q)_2 - (s_p)_2]\} \\ \gamma &= \gamma_{pq} = x^{-1/2} \exp\{2\pi \mathrm{i}[(s_p)_2 + (s_q)_2 - (s_p)_1]\} \end{aligned} \tag{61}$$

then

$$\begin{aligned} &\alpha_{q,Rp} = x/\alpha_{pq} \qquad \beta_{q,Rp} = 1/\gamma_{pq} \qquad \gamma_{q,Rp} = \beta_{pq}/\alpha_{pq} \\ &W_{pq}(1) = \phi(\alpha)\phi(\alpha/\beta) \qquad W_{pq}(2) = \phi(\alpha)\phi(\beta) \\ &\overline{W}_{pq}(1) = \phi(x/\alpha)\phi(x\gamma/\alpha) \qquad \overline{W}_{pq}(2) = \phi(x/\alpha)\phi(1/\gamma) \end{aligned} \tag{62}$$

which is consistent with the relations $\overline{W}_{pq}(n) = W_{q,Rp}(n)$, $W_{pq}(n) = \overline{W}_{q,Rp}(-n)$.

Note that some of these functions are expressed in terms of α, β, and others in terms of α, γ. There is a reason for this. For $N = 3$, we can in principle eliminate the two variables p, q in favour of α, β, or in favour of α, γ, but the result will not necessarily be single-valued. For instance if one tries to expand $W_{pq}(1)$ or $W_{pq}(2)$ in terms of x, α and γ, even to leading order (for x small) one has to solve a quadratic equation for $\exp[2\pi \mathrm{i}(s_p)_1]$, and square roots proliferate in the working.

This difficulty can be traced to the fact that the substitution:

$$p \to M_5 q \qquad q \to M_5 p \tag{63}$$

leaves $\alpha, \gamma, \overline{W}_{pq}(1), \overline{W}_{pq}(2)$ unchanged, but interchanges $W_{pq}(1)$ with $W_{pq}(2)$. Thus $W_{pq}(1)$, $W_{pq}(2)$ cannot be single-valued functions of α and γ and it is useless to look for a convergent expansion with single-valued coefficients (e.g. Laurent polynomials). Note that this objection does not apply to any symmetric function of $W_{pq}(1)$ and $W_{pq}(2)$.

Similarly, the substitution

$$p \to M_3 q \qquad q \to M_3 p \tag{64}$$

leaves $\alpha, \beta, W_{pq}(1), W_{pq}(2)$ unchanged, but interchanges $\overline{W}_{pq}(1)$ with $\overline{W}_{pq}(2)$. Thus neither $\overline{W}_{pq}(1)$ nor $\overline{W}_{pq}(2)$ can be expressed as single-valued functions of α and β.

If $u = \omega^j(\alpha/\beta^2)^{1/3}$, then the three terms on the LHS of (58) are proportional to $W_{pq}(0) = 1, \omega^{-j} W_{pq}(1), \omega^{-2j} W_{pq}(2)$, respectively. Defining the discrete Fourier transform

$$X_{pq}(j) = \sum_{n=0}^{N-1} \omega^{-jn} W_{pq}(n) \tag{65}$$

it follows that

$$X_{pq}(j) = \frac{\Phi(\omega^j u, \omega^{-j} u^2 \beta/x)\Phi(\omega^{j-1} u, \omega^{1-j} u^2 \beta)\Phi(\omega^{j+1} u, \omega^{-1-j} u^2 \beta)}{Q(x)^2 Q(x^3)^4 \tilde{\psi}(\alpha)\tilde{\psi}(\beta)\tilde{\psi}(\alpha/\beta)}. \tag{66}$$

From (20), $X_{pq}(j)/X_{pq}(j-1) = (\omega c_q a_p - \omega^j a_q c_p)/(b_q d_p - \omega^j d_q b_p)$. Interchanging p and q and using (66), it follows that

$$\frac{\omega c_p a_q - \omega^j a_p c_q}{b_p d_q - \omega^j d_p b_q} = \frac{\Phi(\omega^{-j} u, \omega^j x u^2 \beta)\Phi(\omega^{1-j} u, \omega^{j-1} u^2 \beta)}{\Phi(\omega^{-j} u, \omega^j u^2 \beta)\Phi(\omega^{1-j} u, \omega^{j-1} x u^2 \beta)} \tag{67}$$

in agreement with (26).

From (24),

$$\overline{W}_{pq}(1)\overline{W}_{pq}(2) = x h(\alpha/x^2, \beta/x)/[\alpha h(\alpha, \beta)]. \tag{68}$$

Together with (66), this implies that the function f_{pq} defined in (21) is

$$f_{pq} = \frac{V(x)Q(x)^3 Q(x^3)\psi(\alpha)\psi(\beta)\psi(\alpha/\beta)}{h(\alpha,\beta)}. \tag{69}$$

This is consistent with the known results [18, 19]:

$$\begin{aligned} f_{pq} f_{q,Rp} &= 3/k'^{2/3} = x^{-1/3} V(x) Q(x)/Q(x^3)^3 \\ f_{pq}/f_{qp} &= \phi(1/\alpha)\phi(1/\beta)\phi(\beta/\alpha) \end{aligned} \tag{70}$$

and with the automorphisms $M_j^{(1)}, \ldots, R$.

Two further identities that we have obtained but not proved are

$$\frac{\omega a_p b_p c_q d_q - \omega^2 c_p d_p a_q b_q}{a_p b_p c_q d_q - c_p d_p a_q b_q} = -\frac{x h(\alpha/x^2, \beta/x)\phi(x^2\alpha)\phi(x^2\beta)\phi(x^2\beta/\alpha)}{\alpha h(\alpha,\beta)} \tag{71}$$

$$\frac{b_p^N c_q^N - c_p^N b_q^N}{b_p^N d_q^N - c_p^N a_q^N} = \frac{G h(\alpha,\beta) h(x\alpha, \beta/x) h(\alpha/x, x\beta)}{r(\alpha) r(\beta) r(\beta/\alpha)} \tag{72}$$

where the constant G is

$$G = \mathrm{i} x^{1/2} k Q(x)^{-12} Q(x^3)^{-6}. \tag{73}$$

5. Summary

In sections 2 and 3 we have presented some general-N hyperelliptic function identities that are relevant to the chiral Potts model. In section 4 we have considered explicitly the case $N = 3$. Some of the identities of section 4 are special cases of those in section 3. Others are not: for instance we have as yet no generalization to arbitrary N of the result (69) for f_{pq}.

This result is interesting in that it casts light on the analytic nature of f_{pq}: It is indeed a meromorphic function, with only simple poles and zeros. It is not, however, a single-valued function of $a_p, \ldots, d_q$: the automorphism $M_1^{(1)}/M_2^{(1)}$ increments s_1, s_2 respectively by $1, -1$. Applying this to either p or q leaves $a_p, \ldots, d_q$ unchanged, but multiplies or divides f_{pq} by a factor ω. This is consistent with the general formula (3.22) of [20], and (2.44) of [13]. This implies that f_{pq} is a single-valued meromorphic function on an N-fold covering of the $a_p, \ldots, d_q$-surface.

The result (69) answers the problem mentioned on p 3498 of [19], where it is pointed out that a tractable expression for f_{pq} is needed if one is to use the standard inversion relation method [21] to calculate the free energy of the $N = 3$ chiral Potts model, via equation (3.40) of [20]. This has not been done, though the free energy has been calculated by other routes [20, 22, 23]. The problem is similar to that of solving the functional relations for the generalized one-site correlation function, which in turn would yield the order parameters [8, 24].

The identities (15), (17), (53), (55), (57), (58), (66) are proved herein; (62) is proved in [18]; (23) is a corollary of (17); (26), (27), (28) are corollaries of (25); (68) is a specialization of (24); (69) follows from (66) and (68). The remainder, namely (25), (24), (71), (72), are at this stage conjectures, but we believe it should be possible to prove them along the same lines as the proof of (17), or more generally of equation (27) in [18]. Such a proof would use the automorphisms $M_j^{(1)}, \ldots, R$ and locate the poles and zeros of each factor. For $N = 2$ to 4, these identities have all been tested numerically to at least 12 significant digits for arbitrarily chosen values of the parameters.

In a subsequent paper we intend to discuss the hyperelliptic parametrization of the functional relations for the generalized one-site correlation function of the three-state chiral Potts model, and will need the identities (25), (67).

Appendix

The relations (53) follow simply from the identities of Borwein *et al* [16], replacing q therein by either x or x^3. Let us denote equations therein by the prefix BBG. Comparing (BBG 1.6) and (BBG 1.7) with our equations (31)–(51), we see that their functions a, b are

$$a(x) = V(x) \qquad b(x) = V_0(x) + \omega^2 V_1(x) + \omega V_2(x) = V_0(x) - V_1(x). \tag{A1}$$

Hence from (BBG 2.1),

$$c(x^3) = V_1(x). \tag{A2}$$

The third and second of our relations (53) follow at once from the two parts of proposition 2.2 of BBG. Their ratio gives

$$[V_0(x) - V_1(x)]/V_1(x) = Q(x)^3/[3x Q(x^9)^3]. \tag{A3}$$

The LHS of the last equation on p 36 of BBG is $V_0(q)$, so this gives the first of our relations (53). Theorem 2.3 of BBG (which is the main theorem of their paper) becomes

$$[V_0(x) + 2V_1(x)]^3 = [V_0(x) - V_1(x)]^3 + 27x Q(x^3)^9/Q(x)^3$$

i.e.

$$V_1(x)[V_0(x)^2 + V_0(x)V_1(x) + V_1(x)^2] = 3x Q(x^3)^9/Q(x)^3. \tag{A4}$$

Multiplying by (A3), we obtain the last of the relations (53).

Very recently, Hirschhorn [25] has obtained direct and elegant proofs of the identities (53).

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